Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)0119-2
J. Appl. Maths Mechs, Vol. 64, No. 6, pp. 891–894, 2000

0021-8928/00/\$—see front matter

THE SYNTHESIS OF MULTI-PROGRAMME CONTROLS IN BILINEAR SYSTEMS†

N. V. SMIRNOV and T. Ye. SMIRNOVA

St Petersburg

e-mail: nikolay.smirnov@pobox.spbu.ru

(*Received* 13 *July* 1999)

The problem of synthesizing multi-programme controls is considered for a bilinear time-dependent control system with multi-dimensional input. An existence and representation theorem is proved for a control under which the initial system is capable of performing a given set of programmed motions, each of which is asymptotically stable in Lyapunov's sense. © 2001 Elsevier Science Ltd. All rights reserved.

Previous researchers [1, 2] have considered the problem of representing the right-hand sides of systems of differential equations with a prescribed finite family of solutions, as well as the problem of synthesizing controls that realize a given set of programmed motions. Particular attention has been given [2] to the representation of such controls in linear time-independent control systems. Application of the results has been illustrated [1, 2] with reference to the problems of controlling mechanical systems described by Lagrange equations of the second kind and controlling the motion of charged particles in an electromagnetic field.

It will be shown below that this approach can be extended to the class of bilinear control systems, which yield more flexible approximations of non-linear systems than linear systems. One method of constructing bilinear approximations has been proposed [3], and the case of a bilinear time-independent system with one scalar control has been considered [4].

1. FORMULATION OF THE PROBLEM

We consider a bilinear time-dependent control system

$$\dot{\mathbf{x}} = \left(\mathbf{A}(t) + \sum_{i=1}^{r} \mathbf{B}_i(t)u_i\right)\mathbf{x} + \mathbf{F}(t) \tag{1.1}$$

where $\mathbf{x}$ is the n-dimensional phase state vector, $u_1, \ldots, u_r$ are scalar controls, $\mathbf{A}(t)$, $\mathbf{B}_i(t)$ $(i = 1, \ldots, r)$ are real continuous $n \times n$ matrices whose elements are bounded for $t \geqslant 0$ and $\mathbf{F}(t)$ is a real continuous vector function defined for $t \in (-\infty, +\infty)$.

Consider the vector of controls $\mathbf{u} = (u_1, \ldots, u_r)^T$. Let us assume that programmed controls $\mathbf{u}_1(t), \ldots, \mathbf{u}_N(t)$ have been constructed for system (1.1), as well as corresponding programmed motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$. The number N of programmed motions is unrelated to the dimensionality of system (1.1) or to that of the space of controls.

No consideration will be given here to methods of constructing such controls. We merely remark that every programmed control $\mathbf{u}_j(t)$ and programmed motion $\mathbf{x}_j(t)$ are constructed as the solution of a certain boundary-value problem, with different boundary conditions for each $j = 1, \ldots, N$. Thus, substitution of the programmed control $\mathbf{u}_j(t) = (u_{j1}(t), \ldots, u_{jr}(t))^T$ and corresponding programmed motions $\mathbf{x}_j(t)$ into system (1.1) yields an identity with respect to t in the interval where these functions are defined:

$$\dot{\mathbf{x}}_j \equiv \mathbf{P}_j(t)\mathbf{x}_j + \mathbf{F}(t), \quad \mathbf{P}_j(t) = \mathbf{A}(t) + \sum_{i=1}^{r} \mathbf{B}_i(t)u_{ji}(t) \tag{1.2}$$

Problem. It is required to construct a control

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \tag{1.3}$$

†*Prikl. Mat. Mekh.*, Vol. 64, No. 6, pp. 929–932, 2000.

that will implement given programmed motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$ and guarantee their asymptotic stability in Lyapunov's sense.

2. A THEOREM ON THE REPRESENTATION OF A STABILIZING CONTROL

Theorem 1. Suppose the following conditions hold:

1. the programmed motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$ of system (1.1) under controls $\mathbf{u}_1(t), \ldots, \mathbf{u}_N(t)$ are distinct, i.e.

$$\inf_{t \geq 0} \| \mathbf{x}_i - \mathbf{x}_j \| > 0, \quad i \neq j$$

2. the auxiliary linear control systems

$$\dot{\mathbf{y}}_j = \mathbf{P}_j(t)\mathbf{y}_j + \mathbf{Q}_j(t)\mathbf{v}_j \tag{2.1}$$

where

$$\mathbf{Q}_j(t) = (\mathbf{B}_1(t)\mathbf{x}_j(t), \ldots, \mathbf{B}_r(t)\mathbf{x}_j(t)) \tag{2.2}$$

are stabilized by controls

$$\mathbf{v}_j = \mathbf{C}_j(t)\mathbf{y}_j \tag{2.3}$$

where $\mathbf{y}_j = \mathbf{x} - \mathbf{x}_j(t)$, $\mathbf{v}_j = \mathbf{u} - \mathbf{u}_j(t)$ are the deviations from the programmed motions and programmed controls, respectively.

Then a control (1.3) exists that implements the programmed motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$ in such a way that each of them is asymptotically stable in Lyapunov's sense.

Before proceeding to the proof, we present a few previously known definitions and propositions to which we shall refer.

Consider the auxiliary linear control system

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x} + \mathbf{q}(t)u \tag{2.4}$$

The elements of the $n + n$ matrix $\mathbf{P}(t)$ and n-dimensional vector $\mathbf{q}(t)$ are real functions which are continuous for $t \geq 0$, $\mathbf{x}$ is the n-dimensional phase state vector and u is a scalar control.

Theorem 2 [5]. Suppose $\mathbf{P}(t) \in C^{2n-2}_{t\in[0,+\infty)}$, $\mathbf{q}(t) \in C^{2n-1}_{t\in[0,+\infty)}$, $D = \mathbf{P}(t) - \mathbf{E}_n d/dt$ is a differentiation operator and $\mathbf{S} = (\mathbf{q}(t), D\mathbf{q}(t), \ldots, D^{n-1}\mathbf{q}(t))$ is a Lyapunov matrix. Then system (2.4) can be transformed to canonical Frobenius form and the control $u = \mathbf{m}^T \mathbf{x}$ can be so chosen that the closed system

$$\dot{\mathbf{x}} = (\mathbf{P}(t) + \mathbf{q}(t)\mathbf{m}^T)\mathbf{x}$$

is regular (reducible) and has prescribed characteristic indices.

Definition 1 [6]. A square matrix $\mathbf{S}(t)$ is called a Lyapunov matrix if

1. $\mathbf{S}(t) \in C^1_{t\in[0,+\infty)}$
2. the matrices $\mathbf{S}(t)$ and $\dot{\mathbf{S}}(t)$ are bounded for $t \in [0, +\infty)$;
3. the absolute value of the determinant of $\mathbf{S}(t)$ is non-zero, uniformly in $t \geq 0$.

Remark 1. It has been shown [5] that Theorem 2 remains valid for linear systems with r-dimensional control also. The proof of the theorem contains a constructive algorithm for constructing such controls.

Definition 2. A linear control system for which the conditions of Theorem 2 are satisfied will be called stabilizable.

Remark 2. Condition 2 of Theorem 1 means that every system (2.1) is stabilizable in the sense of Definition 2 and that a stabilizing control (2.3) has been constructed.

Proof of Theorem 1. Consider a control (1.3) in the form

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{N} \left(\mathbf{u}_j + \mathbf{C}_j(t)(\mathbf{x} - \mathbf{x}_j) - 2\mathbf{u}_j(t) \sum_{i=1, i\neq j}^{N} \frac{(\mathbf{x}_j - \mathbf{x}_i)(\mathbf{x} - \mathbf{x}_j)}{(\mathbf{x}_j - \mathbf{x}_i)^2} \right) p_j(\mathbf{x}, t) \tag{2.5}$$

where

$$p_j(\mathbf{x},t)=\prod_{i=1,i\neq j}^{N}\frac{(\mathbf{x}-\mathbf{x}_i)^2}{(\mathbf{x}_j-\mathbf{x}_i)^2} \tag{2.6}$$

Control (2.5) and scalar functions (2.6) satisfy the following obvious identities

$$\mathbf{u}(\mathbf{x}_j(t),t)\equiv\mathbf{u}_j(t);\quad p_j(\mathbf{x}_i,t)\equiv 0,\quad i\neq j;\quad p_j(\mathbf{x}_j,t)\equiv 1$$

By virtue of these properties, system (1.1), closed by control (2.5), (2.6), has the given programmed motions $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$, that is, it will move in accordance with one of them provided the appropriate initial data are precisely specified.

We will now prove that the programmed motions of the closed system are asymptotically stable.

Let $u_1(\mathbf{x}, t), \ldots, u_r(\mathbf{x}, t)$ denote the components of the vector $\mathbf{u}(\mathbf{x}, t)$. Consider an arbitrary motion $\mathbf{x}_k(t)$ $(k = 1, \ldots, N)$ and construct the corresponding system for variations. For the variation $\mathbf{y}_k(t) = \mathbf{x}(t) - \mathbf{x}_k(t)$ we obtain the system of equations

$$\dot{\mathbf{y}}_k=\left(\mathbf{A}(t)+\sum_{i=1}^{r}\mathbf{B}_i(t)u_i(\mathbf{y}_k+\mathbf{x}_k,t)\right)\mathbf{y}_k+$$

$$+\left(\sum_{i=1}^{r}\mathbf{B}_i(t)u_i(\mathbf{y}_k+\mathbf{x}_k,t)\right)\mathbf{x}_k-\sum_{i=1}^{r}\mathbf{B}_i(t)\mathbf{x}_k(t)u_i(\mathbf{x}_k,t) \tag{2.7}$$

We single out a linear approximation of system (2.7). To do this we find a representation for the functions $u_j(\mathbf{y}_k - \mathbf{x}_k, t)$. Using the form of control (2.5), we obtain

$$\mathbf{u}(\mathbf{y}_k+\mathbf{x}_k,t)=\sum_{j=1}^{N}(\mathbf{u}_j(t)+\mathbf{C}_j(t)(\mathbf{y}_k+\mathbf{x}_k-\mathbf{x}_j)-2\mathbf{u}_j(t)S_{jk})p_j(\mathbf{y}_k+\mathbf{x}_k,t) \tag{2.8}$$

$$S_{jk}=\sum_{i=1,i\neq j}^{N}\frac{\mathbf{x}_j-\mathbf{x}_i}{(x_j-x_i)^2}(\mathbf{y}_k+\mathbf{x}_k-\mathbf{x}_j)$$

It has been shown [4] that for $j \neq k$ the order of the functions $\rho_j(\mathbf{y}_k + \mathbf{x}_k, t)$ $(j = 1, \ldots, N)$ in the components of the vector $\mathbf{y}_k$ is at least two. Consequently, the expression on the right of (2.8) contains terms linear in $\mathbf{y}_k$ only for $j = k$. Consider the term with $j = k$. After reduction, it can be written as

$$\mathbf{u}_k(t)+\mathbf{C}_k(t)\mathbf{y}_k+\tilde{\mathbf{u}}_k \tag{2.9}$$

$$\tilde{\mathbf{u}}_k=2(\mathbf{C}_k(t)\mathbf{y}_k-2\mathbf{u}_k(t)s_k)s_k+(\mathbf{u}_k(t)+\mathbf{C}_k(t)\mathbf{y}_k-2\mathbf{u}_k(t)s_k)h_k(\mathbf{y}_k)$$

$$s_k=\sum_{i=1,i\neq k}^{N}\frac{\mathbf{x}_k-\mathbf{x}_i}{(\mathbf{x}_k-\mathbf{x}_i)^2}\mathbf{y}_k$$

where we have used the representation of the functions $p_k(\mathbf{y}_k + \mathbf{x}_k, t)$ from [4], and the order of the function $h_k(\mathbf{y}_k)$ in the components of the vector $\mathbf{y}_k$ is at least two.

It is obvious that the order of the components of the vector $\tilde{\mathbf{u}}_k$ in $\mathbf{y}_k$ is also at least two. As a result, Eq. (2.8) becomes

$$\mathbf{u}(\mathbf{y}_k+\mathbf{x}_k,t)=\mathbf{u}_k(t)+\mathbf{C}_k(t)\mathbf{y}_k+\hat{\mathbf{u}}_k \tag{2.10}$$

$$\hat{\mathbf{u}}_k=\tilde{\mathbf{u}}_k+\sum_{j=1,j\neq k}^{N}(\mathbf{u}_j(t)+\mathbf{C}_j(t)(\mathbf{y}_k+\mathbf{x}_k-\mathbf{x}_j)-2\mathbf{u}_j(t)S_{jk})p_j(\mathbf{y}_k+\mathbf{x}_k,t)$$

Using this representation on the right-hand side of system (2.7) and taking notation (1.2), (2.2) into account, after some reduction, we write the system of equations for variations for the programmed motion $\mathbf{x}_k(t)$ in the final form

$$\dot{\mathbf{y}}_k=(\mathbf{P}_k(t)+\mathbf{Q}_k(t)\mathbf{C}_k(t))\mathbf{y}_k+\mathbf{g}_k(\mathbf{y}_k) \tag{2.11}$$

$$\mathbf{g}_k(\mathbf{y}_k)=\left(\sum_{i=1}^{r}\mathbf{B}_i(t)(\mathbf{c}_{ki}(t)\mathbf{y}_k+\hat{u}_{ki})\right)\mathbf{y}_k+\sum_{i=1}^{r}\mathbf{B}_i(t)\mathbf{x}_k(t)\hat{u}_{ki}$$

where $\hat{u}_{ki}$ are the components of the vector $\hat{\mathbf{u}}_k$ and $\mathbf{c}_{ki}(t)\mathbf{y}_k$ is the scalar product of the i-th row of $\mathbf{C}_k(t)$ and the vector $\mathbf{y}_k$; $i = 1, \dots, r$.

By the second condition of Theorem 1, all the systems (2.1) are stabilizable. In that case, by Theorem 2, for all $k = 1, \dots, N$ $r \times n$ matrices $\overline{\mathbf{C}}_k(t)$ exist such that the linear systems

$$\dot{\mathbf{y}}_k = (\mathbf{P}_k(t) + \mathbf{Q}_k(t)\overline{\mathbf{C}}_k(t))\mathbf{y}_k$$

are asymptotically stable in Lyapunov's sense. An algorithm constructing such matrices has been described [5].

Suppose all the matrices $\overline{\mathbf{C}}_k(t)$ ($k = 1, \dots, N$) have been constructed. By the theorem of stability in the first approximation [7] for $\mathbf{C}_k(t) = \overline{\mathbf{C}}_k(t)$, all the systems (2.11) are asymptotically stable in Lyapunov's sense. Consequently, control (2.5), (2.6), where $\mathbf{C}_j(t) = \overline{\mathbf{C}}_j(t)$, guarantees that system (1.1) will have asymptotically stable programmed motions $\mathbf{x}_1(t), \dots, \mathbf{x}_N(t)$. The theorem is proved.

This research was supported financially by the Ministry of Education of the Russian Federation.

REFERENCES

1. ZUBOV, V. I., Interpolation of systems of differential equations. *Dokl. Akad. Nauk SSSR*, 1991, **318**, 1, 28–31.
2. ZUBOV, V. I., Synthesis of multi-programme stable controls. *Dokl. Akad. Nauk SSSR*, 1991, **318**, 2, 274–277.
3. SVORONOS, N. V., STEPHANOPOULOS, G. and ARIS, R., Bilinear approximation of general non-linear dynamic systems with linear inputs. *Int. J. Control*, 1980, **31**, 1, 109–126.
4. SMIRNOV, N. V. and SMIRNOVA, T. Ye., The problem of the stabilization for some programmed motions of the bilinear stationary system. *Proc. 3rd Int. Workshop: Beam Dynamics and Optimization*. BDM, St Petersburg, 1996, 258–263.
5. SMIRNOV Ye. Ya., *Some Problems of Mathematical Control Theory*. Izd. Leningrad.Gos. Univ., Leningrad, 1981.
6. GANTMAKHER, F. R., *Matrix Theory*. Nauka, Moscow, 1988.
7. DEMIDOVICH, B. P., *Lectures on the Mathematical Theory of Stability*. Nauka, Moscow, 1967.

Translated by D.L.